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A New Generalization of the Erdös–Ko–Rado Theorem

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Let $\mathscr{A}$ and $\mathscr{B}$ be systems of k and l element subsets of an n element set respectively. Suppose that $A \cap B \neq \varnothing$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. It is proved that $|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{k-1}\binom{n-1}{l-1}$, whenever $n \geqslant 2k+l-2$ $(k \geqslant l)$.

1. INTRODUCTION

One of the classical theorems in extremal set theory is the following:

THEOREM A. *If $\mathscr{A}$ is a subset system of a finite set $\mathscr{N}$, satisfying the conditions*

$$A_i \not\subset A_j,\ A_i \cap A_j \neq \varnothing,\ |A_i| \leqslant k \quad \text{if} \quad A_i, A_j \in \mathscr{A} \quad (i \neq j)$$

where $2k \leqslant n = |\mathscr{N}|$ then $|\mathscr{A}| = m \leqslant \binom{n-1}{k-1}$ (see [1]).

This theorem of Erdös, Ko, and Rado has been generalized by several authors. The following generalization is due to Kleitman [2].

THEOREM B. *If $\mathscr{A}$ and $\mathscr{B}$ are subset systems of a finite set $\mathscr{N}$ satisfying the conditions*

$$|A_i \cap B_r| \neq 0,\ |A_i| = k,\ |B_r| = l \quad \text{for} \quad A_i \in \mathscr{A} \text{ and } B_r \in \mathscr{B},$$

where $k + l \leqslant n = |\mathscr{N}|$ and $|\mathscr{A}| \geqslant \binom{n-1}{k-1}$ then $|\mathscr{B}| \leqslant \binom{n-1}{l-1}$.

Actually in [2], Kleitman proves a stronger theorem.

Among the most useful tools in extremal set theory, we have the Kruskal–Katona theorem. As Daykin observed, Theorem A follows easily from it (see [3]).

Let $\mathscr{A}$ be a k-uniform subset system of $\mathscr{N}$. $\mathscr{A}_{k-1}$ denotes the family of subsets C with $|C| = k-1$ and $C \subset A$ for some $A \in \mathscr{A}$. If $|\mathscr{A}| = m$ is fixed,

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then the Kruskal–Katona theorem gives us a difficult formula for the minimum of $|\mathscr{A}_{k-1}|$. But we also have an optimal construction (see [4]).

Let us fix an order $v_1, \ldots, v_n$ of $\mathscr{N}$.

DEFINITION. We call a subset system $\mathscr{A}$ *first* with $|\mathscr{A}| = m$ if it is first among the m element subset systems (we are talking about k uniform systems) with respect to the lexicographic order, i.e., if it consists of the first m k-sets. (Here $(v_{i1}, \ldots, v_{ik}) < (v_{j1}, \ldots, v_{jk})$ iff for some d and for all $q > d$ $i_q = j_q$ and $i_d < j_d$). So the first $\binom{n-1}{k-1}$ k-sets are exactly the k-subsets of $(v_1, \ldots, v_{n-1})$.

THEOREM C. *Among the k uniform subset systems* $\mathscr{A}$ *in* $\mathscr{N}$ *with* $|\mathscr{A}| = m$ *fixed,* $|\mathscr{A}_{k-1}|$ *is minimal for the first system.*

We shall use this theorem and the method of cyclic permutations to prove the following new theorem:

THEOREM. *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are subset systems of a finite set* $\mathscr{N}$, *satisfying the conditions*

$$A_i \cap B_r \neq \varnothing,\ A_i \not\subset A_j,\ B_r \not\subset B_s,\ |A_i| \leqslant k,\ |B_r| \leqslant l$$
$$\text{for } A_i, A_j \in \mathscr{A} \text{ and } B_r, B_s \in \mathscr{B}$$

then

(1) *if* $k > l$, $2k + l - 2 \leqslant n$ *then* $|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{k-1}\binom{n-1}{l-1}$,

(2) *if* $k = l$, $2k \leqslant n$ *then* $|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{k-1}^2$.

2. PROOF OF THE FIRST PART

LEMMA 2.1. *It is enough to prove the theorem for k- and l-uniform systems.*

Proof. We make use of the well-known Sperner operation. Let $k > r = \min(|A| \mid A \in \mathscr{A})$, $\mathscr{R} = (A \mid \in \mathscr{A},\ |A| = r)$, $\mathscr{R}^{r+1} =$ (the $r+1$ element sets in $\mathscr{N}$ containing some elements of $\mathscr{R}$). It easily follows that $\mathscr{A}' = (\mathscr{A} \backslash \mathscr{R}) \cup \mathscr{R}^{r+1}$ is a Sperner system and for $A' \in \mathscr{A}'$, $B \in \mathscr{B}$, $A' \cap B \neq \varnothing$. By a trivial lemma of Sperner (see [5]) $|\mathscr{R}^{r+1}| \geqslant ((n-r)/(r+1))|\mathscr{R}| \geqslant |\mathscr{R}|$. Replacing $\mathscr{R}$ by $\mathscr{R}^{r+1}$ in $\mathscr{A}$ is the Sperner operation. Applying the operation for $\mathscr{A}$ and $\mathscr{B}$ iteratively we get the uniform systems $\mathscr{A}^*$ and $\mathscr{B}^*$ satisfying the conditions of our theorem with $|\mathscr{A}|\,|\mathscr{B}| \leqslant |\mathscr{A}^*|\,|\mathscr{B}^*|$. ∎

From now on $\mathscr{A}$ and $\mathscr{B}$ will be k- and l-uniform system, respectively.

LEMMA 2.2. *It is enough to prove the theorem when* $\bar{\mathscr{A}} = (\bar{A} \mid A \in \mathscr{A})$ (or $\bar{\mathscr{B}} = (\bar{B} \mid B \in \mathscr{B})$) *is first.*

Proof. Let $(\bar{\mathscr{A}})_l = (X \mid X \subset \mathscr{N},\ |X| = l\ \exists A \in \bar{\mathscr{A}}\colon X \subset A)$ $(l \leqslant n-k)$. If we have $\mathscr{A}$ fixed, then the maximal $\mathscr{B}$ consists of the l-sets not in $(\bar{\mathscr{A}})_l$. If $|\mathscr{A}| = |\bar{\mathscr{A}}|$ is fixed then by Theorem C, $(\bar{\mathscr{A}})_l$ is minimal if $\bar{\mathscr{A}}$ is first. So we can reach the maximum of $|\mathscr{A}|\ |\mathscr{B}|$ with $\bar{\mathscr{A}}$ being first. The same is true for $\bar{\mathscr{B}}$. ∎

LEMMA 2.3. *If* $|\mathscr{A}| \geqslant \binom{n-1}{k-1}$ *and* $\bar{\mathscr{A}}$ *is first then* $v_n \in B$ *for all* $B \in \mathscr{B}$.

Proof. The first $\binom{n-1}{k-1} = \binom{n-1}{n-k}$ $(n-k)$-sets are all the $(n-k)$ subsets of $\mathscr{N} \setminus \{v_n\}$ and their subsets are not in $\mathscr{B}$. All the others contain v_n.

Proof of Part (1)

If $|\mathscr{A}|\ |\mathscr{B}| \geqslant \binom{n-1}{k-1}\binom{n-1}{l-1}$ then either $|\mathscr{A}| \geqslant \binom{n-1}{k-1}$ or $|\mathscr{B}| \geqslant \binom{n-1}{l-1}$. Let us suppose that $|\mathscr{A}| \geqslant \binom{n-1}{k-1}$, $\bar{\mathscr{A}}$ is first so $v_n \in B$ for all $B \in \mathscr{B}$. We denote the set of pairs $((A, B) \mid A \in \mathscr{A},\ B \in \mathscr{B})$ by $\mathscr{A} \times \mathscr{B}$. It suffices to prove that $\binom{n-1}{k-1}\binom{n-1}{l-1} = |\mathscr{A}' \times \mathscr{B}'| \geqslant |\mathscr{A} \times \mathscr{B}| = |\mathscr{A}|\ |\mathscr{B}|$, where $\mathscr{A}'$ and $\mathscr{B}'$ consist of the k- and l- sets in $\mathscr{N}$ containing v_n, respectively. Let us define $C_m = ((A, B) \mid |A \cap B| = m,\ (A, B) \in \mathscr{A} \times \mathscr{B})$ and C'_m with a similar definition for $\mathscr{A}' \times \mathscr{B}'$. Now we have $\mathscr{A} \times \mathscr{B} = \bigcup_1^l C_m$ and $\mathscr{A}' \times \mathscr{B}' = \bigcup_1^l C'_m$. It suffices to prove that $|C_m| \leqslant |C'_m|$ for $m = 1, \ldots, l$.

DEFINITION. Let us consider n equidistant points on a circle. By a *cyclic permutation* π *of the set* $\mathscr{N}$ we mean an arrangement of its elements at the points, where two arrangements represent the same permutation if we might obtain one from another by turning the circle. The set of all cyclic permutations of $\mathscr{N}$ is denoted by $\mathscr{P}_{\mathscr{N}}$.

DEFINITION. k consecutive elements of $\mathscr{N}$ in $\pi \in \mathscr{P}_{\mathscr{N}}$ are called a *k-block* of π. In the case $k < n$ the meaning of the notion *endpoints of a block* is clear. We call the clockwise and counter-clockwise directions negative and positive, respectively.

DEFINITION. A pair $/A, B/ \in C_m$ $/m = 1, \ldots, l/$ has a *representation* in $\pi \in \mathscr{P}_{\mathscr{N}}$ if A and B are blocks in π and B contains the negative endpoint of the block A.

By simple counting we obtain that a pair $(A, B) \in C_m$ has exactly $(n-k-l-m)!(k-m)!(l-m)!m!$ representations in the permutations $\pi \in \mathscr{P}_{\mathscr{N}}$. This number is the same for all pairs in C_m or C'_m. So it suffices to prove that the number of representations of pairs in C_m is less than or equal to the same number for C'_m in each $\pi \in \mathscr{P}_{\mathscr{N}}$ $(m = 1, \ldots, l)$.

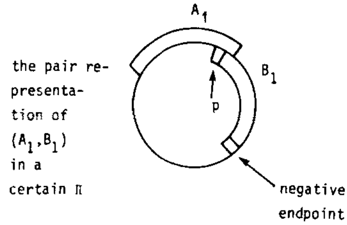

FIGURE 1

LEMMA 2.4. *If* $(A_i, B_i) \in C_m$ $(i=1,..., t)$ *are represented in* $\pi \in \mathcal{P}_{\mathcal{N}}$ *then* $\bigcap_{i=1}^{t} (A_i \cap B_i) \neq \emptyset$.

Proof. v_n is an element of all the blocks B_i. $2k+l-2 \leqslant n$ and $k > l$, so $2l+k-2 \leqslant n$ as well, so we have a *negative extreme* block B_1 which contains the negative endpoints of all the blocks B_i. Now the positive endpoint p of B_1 is, of course, in the intersection of the blocks B_i. If $p \notin A_i$ for some $i \leqslant t$, then A_i contains the negative endpoint of B_1, for B_1 and A_i are blocks in π with $A_i \cap B_1 \neq \emptyset$, so A_i, B_i, and B_1 cover the set $\mathcal{N}$, a contradiction $(2l+k-2 \leqslant n)$. ∎

See Figs. 1 and 2.

Let us consider a certain $\pi \in \mathcal{P}_{\mathcal{N}}$ and the blocks A_i' and B_i' containing v_n with $|A_i' \cap B_i'| = m$. From the definition of C_m' we know that the pairs (A_i', B_i') are exactly the pairs in C_m' having representation in π.

By Lemma 2.4 there are at least as much representations of pairs in C_m' in any π as of pairs in C_m.

For $|\mathcal{B}| \geqslant \binom{n-1}{l-1}$ and $\bar{\mathcal{B}}$ first we might prove the theorem in the same way. ∎

3. PROOF OF THE SECOND PART

By Lemmas 2.1, 2.2 we might suppose that $\mathcal{A}$ and $\mathcal{B}$ are k uniform systems and $\bar{\mathcal{A}}$ is first, where $|\mathcal{A}| \geqslant |\mathcal{B}|$.

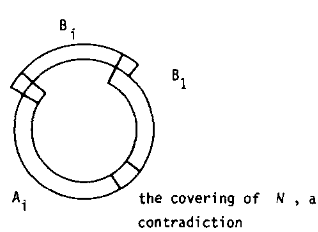

FIGURE 2

LEMMA 3.1. *Under the conditions of our theorem (part (2))* $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{k}$ *holds.*

Proof. Let us consider the cyclic permutations of $\mathscr{N}$. Among the $(n-1)!$ cyclic permutations there are exactly $k!(n-k)!$ in which a certain k-subset of $\mathscr{N}$ forms a block of consecutive elements. If $A \in \mathscr{A}$ forms a block in π then the k elements of $\mathscr{N}$ neighbouring A from negative side do not correspond to any element B of $\mathscr{B}$. So if we have $t \leqslant n$ blocks corresponding to elements of $\mathscr{A}$, then we also have at least t blocks which do not correspond to elements of $\mathscr{B}$, i.e., the number of blocks formed by elements of $\mathscr{A}$ and $\mathscr{B}$ (we count them with multiplicity) is at most n in any $\pi \in \mathscr{P}_{\mathscr{N}}$. So we have $(n-1)!\, n \geqslant k!\,(n-k)!\,(|\mathscr{A}| + |\mathscr{B}|)$ by counting all the blocks in the permutations $\pi \in \mathscr{P}_{\mathscr{N}}$, formed by elements of $\mathscr{A}$ and $\mathscr{B}$. ∎

LEMMA 3.2. *If* $|\mathscr{A}| \geqslant \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{n-t-1}{k-1}$ *for* $t = (1,..., k)$ *then, supposing* $\bar{\mathscr{A}}$ *is first, we have* $(v_n,..., v_{n-t+1}) \subset B \in \mathscr{B}$ *and* $|\mathscr{B}| \leqslant \binom{n-t}{k-t}$.

Proof. The first $\binom{n-1}{k-1} + \cdots + \binom{n-t}{k-1} = \binom{n-1}{k-1} + \cdots + \binom{n-t}{k-t}$ $(n-k)$-sets (which are in $\bar{\mathscr{A}}$) contain all the k element sets not containing $(v_n,..., v_{n-t+1})$ and as simple counting shows $|\mathscr{B}| \leqslant \binom{n-t}{k-t}$. ∎

LEMMA 3.3. *If* $|\mathscr{A}| \geqslant \binom{n-1}{k-1} + \binom{n-2}{k-1}$ *then the second statement of our theorem is valid.*

Proof. Let $\binom{n-1}{k-1} + \cdots + \binom{n-t}{k-t} \leqslant |\mathscr{A}| \leqslant \binom{n-1}{k-1} + \cdots + \binom{n-t-1}{k-1}$. If $\bar{\mathscr{A}}$ is first then by Lemma (3.2) we have $|\mathscr{B}| \leqslant \binom{n-t}{k-t} \leqslant 2^{1-t}\binom{n-1}{k-1}$ using $2k \leqslant n$. We know that $|\mathscr{A}| \leqslant (t+1)\binom{n-1}{k-1}$ and for $t \geqslant 3$ $2^{t-1} \geqslant t+1$ holds. It implies $|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{k-1}^2$ for $t \geqslant 3$. For $t = 2$ $|A| \leqslant \binom{n-1}{k-1}(1 + \frac{n-1}{n-1} + \frac{n-k-1}{n-1})$ and $|B| \leqslant \frac{k-1}{n-1}\binom{n-1}{k-1}$. It suffices to show that $(k-1)(3n-2k-2) \leqslant (n-1)^2$. Let us define r by $n = 2k + r$. Easy computation shows that the above inequality is equivalent to

$$1 \leqslant 2k + rk + r^2 + r.$$

This follows from $r \geqslant 0$ and $k \geqslant 1$. ∎

Proof of Part (2)

Using the above lemmas we might suppose that $\binom{n-1}{k-1} \leqslant |A| < \binom{n-1}{k-1} + \binom{n-2}{k-1}$, $\bar{A}$ is first, $v_n \in B$ for all $B \in \mathscr{B}$. All the $(n-k)$-sets not containing v_n are in $\bar{A}$ so all the k-sets containing v_n are in A.

$\bar{\mathscr{A}}$ is first so the sets in $\mathscr{A}$ not containing v_n contain v_{n-1} for $|\mathscr{A}| < \binom{n-1}{k-1} + \binom{n-2}{k-1}$. Let us omit the sets in $\mathscr{A}$ containing v_n. If we delete v_{n-1} from the remaining ones, we obtain a $(k-1)$ uniform subset system $\mathscr{A}_0$ of $(v_1,..., v_{n-2})$. We have $|\mathscr{A}| = |\mathscr{A}_0| + \binom{n-1}{k-1}$. Omitting the sets in $\mathscr{B}$ that contain both v_n and v_{n-1} and deleting v_n from the remaining ones we obtain a

$(k-1)$ uniform subset system $\mathscr{B}_0$ of $(v_1,\ldots, v_{n-2})$. We have $|\mathscr{B}| \leqslant |\mathscr{B}_0| + \binom{n-2}{k-2}$ and of course $|\mathscr{B}_0| \leqslant \binom{n-2}{k-1}$.

Now $|\mathscr{A}|\,|\mathscr{B}| \leqslant (|\mathscr{A}_0| + \binom{n-1}{k-1})(|\mathscr{B}_0| + \binom{n-2}{k-2}) = \binom{n-1}{k-1}\binom{n-2}{k-2} + |\mathscr{B}_0|\binom{n-1}{k-1} + |\mathscr{A}_0|(|\mathscr{B}_0| + \binom{n-2}{k-2}) \leqslant \binom{n-2}{k-2}\binom{n-1}{k-1} + (|\mathscr{A}_0| + |\mathscr{B}_0|)\binom{n-1}{k-1}$. For the systems $\mathscr{A}_0$ and $\mathscr{B}_0$ we have $A_0 \cap B_0 \neq \varnothing$ if $A_0 \in \mathscr{A}_0$, $B_0 \in \mathscr{B}_0$ (it follows easily from the definitions). Applying lemma (3.1) we obtain $(|\mathscr{A}_0| + |\mathscr{B}_0|) \leqslant \binom{n-2}{k-1}$ so $|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{k-1}^2$.

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